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Author(s): Salvador Barbera

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A NOTE ON GROUP STRATEGY-PROOF DECISION SCHEMES

BY SALVADOR BARBERA¹

DECISION SCHEMES ARE SOCIAL decision-making procedures which assign a lottery on alternatives to each N -tuple of rankings over alternatives. A decision scheme is individual strategy-proof if no individual expected utility maximizer ever finds it advantageous to manipulate its outcome by misrepresenting his preference ranking of alternatives. Gibbard [2] has recently characterized the class of all decision schemes which meet this requirement. It is a large class; some interesting subclasses of it are studied in [1]. This note defines a stronger property of decision schemes, that of group strategy-proofness, and characterizes the class of decision schemes which satisfy this property.

The reader is referred for notation and definitions to Gibbard's work [2]. Some additional definitions are introduced below.

DEFINITION 1: A *unilateral scheme* is called strict iff it is neither constant nor duple nor the sum of a constant and a duple scheme.

DEFINITION 2: A *duple scheme* is called strict iff it is not constant.

DEFINITION 3: If d is a unilateral scheme and individual i is the sole determiner of the value of d , we say that d is *i -unilateral*. If d is a duple scheme and the only two alternatives which are assigned nonzero probabilities under d are x and y , then we say that d is *xy -duple*.

DEFINITION 4: Given a decomposition of a decision scheme as the sum of nonperverse and localized schemes, each of which is either unilateral or duple, in the form

$$(1) \quad d = d_0 + d_1 + \dots + d_m,$$

we define the *reduced form* of this decomposition of d to be the decomposition

$$(2) \quad d = d'_0 + d'_1 + \dots + d'_n$$

obtained from (1) by performing the following operations in the indicated order:

- (i) For each d_i , let \hat{d}_i be the constant scheme that assigns to each alternative its minimal value under d_i . Express each d_i as the sum of \hat{d}_i and $d_i - \hat{d}_i$.
- (ii) In the resulting decomposition, add all constant schemes together.
- (iii) Add all remaining i -unilateral schemes together, for each individual i .
- (iv) Add all remaining xy -duple schemes together, for each pair of alternatives x, y .

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Notice that if a unilateral scheme resulting from (iii) is also duple, then it will be among the schemes considered in step (iv).

Proofs of the following four Propositions are left to the reader.

PROPOSITION 1: *The sum of i-unilateral schemes is an i-unilateral scheme.*

PROPOSITION 2: *The sum of an i-unilateral duple scheme and a strict i-unilateral scheme is strict i-unilateral.*

PROPOSITION 3: *The sum of xy-duple schemes is an xy-duple scheme.*

PROPOSITION 4: *The reduced form of a decomposition of d is such that:*

(4.1) *At most one constant scheme appears in the decomposition.*

(4.2) *At most one strict i-unilateral scheme appears in the decomposition for each individual i.*

(4.3) *At most one strict xy-duple scheme appears in the decomposition for each pair of alternatives x, y.*

(4.4) *There cannot be two duple i-unilateral schemes in the decomposition.*

(4.5) *If the xy-duple component of the decomposition is i-unilateral, there is no strict i-unilateral component.*

(4.6) *All components are localized and non-perverse.*

DEFINITION 5: A decision scheme *d* is *group manipulable* iff there exists a ranking *N*-tuple **P**, a set of individuals $\{i_1, i_2, \dots, i_h\}$, *h* preference rankings $P'_{i_1}, P'_{i_2}, \dots, P'_{i_h}$, and *h* utility scales $u_{i_1}, u_{i_2}, \dots, u_{i_h}$ which fit $P_{i_1}, P_{i_2}, \dots, P_{i_h}$, respectively, in such a way that

$$u_{ij}(dP'') > u_{ij}(dP) \quad \text{for all } j \in \{i_1, i_2, \dots, i_h\},$$

when **P''** is the ranking *N*-tuple defined by

$$P''_{ij} = P'_{ij} \quad \text{for } j \in \{i_1, \dots, i_h\},$$

$$P''_{ij} = P_{ij} \quad \text{for } j \notin \{i_1, \dots, i_h\}.$$

A decision scheme is *group strategy-proof* iff it is not group manipulable.

PROPOSITION 5: *If there exist profiles **P** and **P'**, individuals *i* and *j*, and alternatives *x, y, z, w*, such that (i) $\{x, y\} \neq \{z, w\}$; (ii) $P_k = P'_k$ for $k \notin \{i, j\}$; (iii) $xP_i!y$ & wP_jz ; (iv) $zP_j!w$ & yP_jx ; and (v) $d(y, P') - d(y, P) = d(x, P) - d(x, P') > 0$ & $d(w, P') - d(w, P) = d(z, P) - d(z, P') > 0$ & $d(r, P') = d(r, P)$ for all $r \notin \{x, y, z, w\}$; then *d* is group manipulable.*

PROOF: It suffices to consider utility scales u_i and u_j fitting P_i and P_j , respectively, in such a way that

$$[u_i(w) - u_i(z)][d(w, P') - d(w, P)] > [u_i(x) - u_i(y)][d(y, P') - d(y, P)]$$

while

$$[u_j(y) - u_j(x)][d(y, \mathbf{P}') - d(y, \mathbf{P})] > [u_j(z) - u_j(w)][d(w, \mathbf{P}') - d(w, \mathbf{P})].$$

Notice that this choice is always possible since $xP_i!y$ and $zP_j!w$. Then, individuals i and j endowed with utility scales u_i and u_j would find it advantageous to form a coalition and manipulate from \mathbf{P} to \mathbf{P}' .

THEOREM: *A decision scheme d is group strategy-proof iff it is localized, nonperverse, and either unilateral or the sum of a constant and a duple scheme.*

PROOF: The sufficiency part of the theorem is obvious. To prove necessity, suppose d is group strategy-proof. Then it is individual strategy-proof and, by Gibbard's theorem [2], it is the sum of nonperverse and localized schemes, each of which is either unilateral or duple, in the form $d = d_0 + d_1 + \dots + d_m$.

Assume without loss of generality that this decomposition is in reduced form. The result is trivial when the number of alternatives equals two. The following argument applies when there are at least three alternatives.

Step 1: The decomposition of d cannot involve two strict unilateral schemes.

Suppose there were two such schemes, d_i and d_j . By (4.2), $i \neq j$. There would exist \bar{P}_i, \bar{P}_j and alternatives $\{x, y\} \neq \{z, w\}$ such that

$$(3) \quad x\bar{P}_i!y \quad \text{and} \quad \varepsilon_i^y(d_i, \mathbf{P}) = \varepsilon_i^x(d_i, \mathbf{P}/P_i^y) \neq 0 \quad \text{for any } \mathbf{P} \text{ such that } P_i = \bar{P}_i,$$

$$(4) \quad z\bar{P}_j!w \quad \text{and} \quad \varepsilon_j^w(d_j, \mathbf{P}) = \varepsilon_j^z(d_j, \mathbf{P}/P_j^w) \neq 0 \quad \text{for any } \mathbf{P} \text{ such that } P_j = \bar{P}_j.$$

Without loss of generality, suppose $w\bar{P}_i z$ and $y\bar{P}_j x$. Then let \mathbf{P} be such that $P_i = \bar{P}_i, P_j = \bar{P}_j$. Consider \mathbf{P}' such that

$$[h \notin \{i, j\}] \rightarrow [P_h = P'_h], \quad P'_i = P_i^y, \quad P'_j = P_j^w.$$

Since all components of the reduced decomposition of d are nonperverse and localized, the conditions in Proposition 5 would hold, contradicting the fact that d is group strategy-proof.

Step 2: The decomposition of d cannot involve at the same time a strict unilateral scheme and a strict duple scheme. For, suppose d_i, d_{zw} were such schemes. There would then exist \bar{P}_i, \hat{P}, P'_j such that

$$(5) \quad x\bar{P}_i!y \quad \text{and} \quad \varepsilon_i^y(d_i, \mathbf{P}) = \varepsilon_i^x(d_i, \mathbf{P}/P_i^y) \neq 0 \quad \text{for any } \mathbf{P} \text{ such that } P_i = \bar{P}_i,$$

$$(6) \quad d_{zw}(z, \hat{P}) \neq d_{zw}(z, \hat{P}/P'_j).$$

It follows from (4.5) and the fact that d_{zw} is duple that \hat{P} can be chosen in such a way that $\hat{P}_i = \bar{P}_i, j \neq i, z\hat{P}_j!w, P'_j = \hat{P}_j^w$, and $y\hat{P}_j x$. Suppose without loss of generality that $w\bar{P}_i z$. Consider \hat{P}' such that

$$[h \notin \{i, j\}] \rightarrow [\hat{P}'_h = \hat{P}_h], \quad \hat{P}'_i = \hat{P}_i^y \quad \text{and} \quad \hat{P}'_j = \hat{P}_j^w.$$

Since all components of the reduced decomposition of d are nonperverse and localized, the conditions of Proposition 5 would hold for \hat{P} and \hat{P}' , contradicting the fact that d is group strategy-proof.

Step 3: The decomposition of d cannot involve at the same time two strict duple schemes. For, suppose d_{xy} and d_{zw} were such schemes. Since we are working with a reduced decomposition, and by (4.4), there would exist $\hat{P}, \hat{P}, i \neq j, P'_i$ and P'_j such that

$$(7) \quad d_{xy}(x, \hat{P}) \neq d_{xy}(x, \hat{P}/P'_i), \quad \text{and}$$

$$(8) \quad d_{zw}(z, \hat{P}) \neq d_{zw}(z, \hat{P}/P'_j).$$

Suppose without loss of generality that $x\hat{P}_iy, w\hat{P}_iz, y\hat{P}_jx,$ and $z\hat{P}_jw$. Then let P be such that

$$(\forall h)[(xP_hy \leftrightarrow x\hat{P}_hy) \ \& \ (zP_hw \leftrightarrow z\hat{P}_hw) \ \& \ xP_i!y \ \& \ zP_j!w].$$

Let P' be such that

$$[h \notin \{i, j\}] \rightarrow [P'_h = P'_h], \quad P'_i = P'_i^y \quad \text{and} \quad P'_j = P'_j^w.$$

Again, application of Proposition 5 would lead to a contradiction.

Step 4: By Steps 1, 2, and 3, we know that d is group strategy-proof only if its reduced decomposition consists of at most a constant term and another term, which must in turn be localized, nonperverse, and either unilateral or duple. The fact that the sum of a constant and a unilateral scheme is unilateral completes the proof.

Finally, notice that only two-individual coalitions need to be used in the proof of the theorem. Thus, the fact that coordination of large groups of individuals for strategic purposes might be costly and/or easy to detect will not help much in dispelling the negative impact of the result presented here.

Universidad de Bilbao

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